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SOLUTION OF THE EQUATION OF VERTICAL FLIGHT OF A ROCKET BY THE GRAPHIC-MATHEMATICAL METHOD

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SOLUTION OF THE EQUATION FOR VERTICAL FLIGHT OF A ROCKET ACCORDING TO THE GRAPHICAL-MATHEMATICAL METHOD

M E X I C O

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SOLUTION OF THE EQUATION FOR VERTICAL FLIGHT OF A ROCKET IN THE ATMOSPHERE

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INTRODUCTION

The research topic entitled "SOLUTION OF THE EQUATION FOR VERTICAL FLIGHT OF A ROCKET ACCORDING TO THE GRAPHICAL-MATHEMATICAL METHOD" is one of many studies undertaken on the design and construction of SCT rockets using solid and liquid fuels, for the Secretariat of Communications and Transportation.

All these personal studies will be published when the writer has the time to do so since they represent scientific and technical studies on rockets in accordance with the limited resources available to us.

SOLUTION OF THE EQUATION FOR VERTICAL FLIGHT

OF A ROCKET IN THE ATMOSPHERE

C H A P T E R I

General Considerations

To determine the flight characteristics of SCT-1 and SCT-2 rockets /1 * the point by point calculation of the trial and error method was used until the flight equation was satisfied. This is a quick method when we have an idea of the rocket's behaviour. But if we wish to follow a logical, mathematical order, then we must solve the differential flight equation within finite boundaries.

In order to facilitate study, we shall consider only the rocket's vertical motion in the atmosphere; in another study we shall discuss the more general case of the rocket moving horizontally.

Newton's laws are the basis of this study since they explain the phenomena to which bodies at rest and in motion are subjected.

Newton's Second Law. The rate of change of momentum of a body is directly proportional to the acting force and the momentum takes place in the direction in which the force acts. The equation is $F = Ma$. From this equation we can also say that the acceleration produced by the force applied to the body is directly proportional to the magnitude of the force and inversely proportional to the mass of the body.

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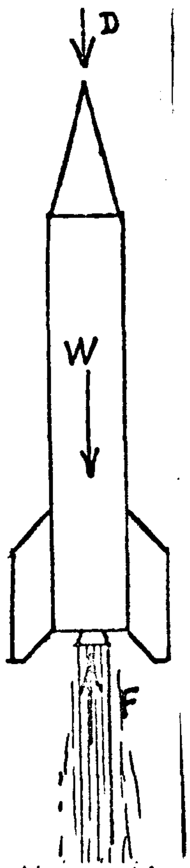
* Numbers [illegible]

In Figure 1 a rocket is shown as a body on which various forces /2 act with the result that motion in full flight is affected. The forces acting on the rocket are:



F: The propelling force which produces the vertical acceleration motion from the gases escaping from the exhaust nozzle.

W: Force due to the earth's attraction, called gravity. The value assigned to it is M x g in which M is the mass at the time the mathematical analysis is conducted, and g is the gravitational acceleration. This force tends to prevent the rocket from rising.

D: The resistance force of the air, within the atmosphere which also hinders the free movement of the rocket in the atmosphere. In Spanish this is also called Aerodynamic or Retarding Resistance; in English the word "drag" is preferred.

At any time when the propelling force F of the combustion gases acts, the effective force Fe which produces the rocket's acceleration $\underline{a}$ is assigned a value according to the second principle of Newton's Law.

$$Fe = Ma = F - W - D \qquad (1)$$

Later we shall examine equation (1) for all values given at time t after the rocket's ignition.

The component parts of a jet-propelled rocket are:

1. The propulsion system.
2. The mass of propellants (the fuel plus the oxidizer).
3. Dead space of the rocket (body).

4. The payload (explosives or material to be transported).

During propelled flight the total mass decreases. If M_o is the mass of the missile before ignition, m_p the mass of the propellants (fuel plus the oxidizer), and m the mass of the body and payload, then we have:

During flight, the mass of the propellants progressively decreases. /3

We shall assume in this study that the consumption per unit of time is constant which means that the combusion gases will leave at the same velocity - V_j; we shall discuss this velocity in greater detail later. V_j varies according to altitude for equal rates of consumption for fuel and oxidizer per unit of time:

If we make:

w = propellant consumption per second,

x(%) = the percentage or fraction of total propellant mass consumed per unit of time, then we have:

$$X m_p = \frac{w}{g}$$

$$w = g \times m_p \qquad \text{(weight/second)}$$

$$-\frac{dM}{dt} = x\, m_p$$

At any time t, mass M of the rocket will be:

$$M = M_o - x\, m_p t \qquad \text{(slug in the US.) (2)}$$

If we differentiate equation (2), then we get the first derivative which is the relationship showing how the mass of the rocket varies with respect to time.

$$M_o = m + m_p \qquad \text{(slug/unit of time) (3)}$$

The negative sign indicates that the mass decreases as time goes on.

CHAPTER II

PROPELLING FORCE

The thrust F of a rocket is the reaction experienced by a body due to the action of high velocity of the material; in this case, the exhaust gases.

In dynamics the word "momentum" is defined as the product of the mass times its velocity; In the case of the gases escaping at high speed from the exhaust nozzle, the momentum of many uniform particles streaming at a uniform velocity. Thus, the gas stream can be expressed in the momentum of a solid body of the same mass and velocity.

When the pressure of the medium is equal to the pressure of the gases escaping from the exhaust nozzle, this is called thrust and is expressed as follows:

$$F = \frac{\dot{w}}{g} v = \frac{\dot{w}}{g} V_j .$$

Nowadays it is possible to get a high combustion efficiency with respect to the ideal heat of the chemical reaction, with values from 94 to 99% being attained.

The internal efficiency or the relation between the kinetic energy at the exhaust nozzle outlet and the thermal energy of the chemical reaction is defined as:

$$n_{int} = \frac{\frac{1}{2} \left(\frac{\dot{w}}{g}\right) V_j^2}{\dot{w}\, Q_R\, J} = \frac{V_j^2}{2 g J Q_R} \qquad (5)$$

in which n_{int} is the internal efficiency,

V_j the effective velocity of the escaping gases

Q_R the reaction heat per unit of propellant

J the mechanical heat equivalent.

The propulsion efficiency will be:

If V is the velocity of the rocket

$$n_p = \frac{FV}{FV + \frac{1}{2}\,\frac{w}{g}\,(V_j - V)^2} = \frac{\frac{2V}{V_j}}{1 + \left(\frac{V}{V_j}\right)^2}$$

The total efficiency will be:

$$n_t = \frac{FV}{w\,Q_R\,J + \frac{1}{2}\,\frac{w}{g}\,V^2} = \frac{V_j\,V}{g\,Q_R J + \frac{V^2}{g}}$$

$$n_t = \frac{\frac{2V}{V_j}}{\frac{1}{ni_{h_t}} + \left(\frac{V}{V_j}\right)^2} \qquad (7)$$

The specific impulse, also known as specific thrust, is defined by the following formula:

$$I_s = \frac{F}{w} = \frac{V_j}{g} \qquad (8)$$

It is the force exerted by a unit of propellant per second.

We have presented the foregoing considerations in order to indicate clearly how the maximum propellant force can be atteined;nevertheless, in summarizing matters on efficiencies and atmospheric conditions, the following factors must be taken into consideration:

In the combustion chamber there are losses due to:

improper mixing causing incomplete combustion (1%);

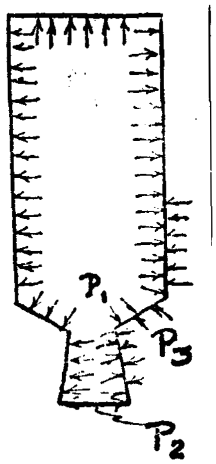
loss of heat through surrounding surfaces (2%);

energy lost through the exhaust (57 to 27%);

loss of residual energy in the exhaust gases (20%).

It is possible to deduce from this that from 0 to 50% of the rocket's propellant energy can be utilized.

With regard to the variation in propellant force due to altitude or variation in atmospheric pressure, we use average velocities in our calculations, but in order to have an idea of how these variations occur, we shall advance the following theory:



If p_1 = the internal pressure of the combustion chamber, as well as the pressure in the throat;

p_2 = pressure at the exhaust nozzle of the combustion chamber;

p_3 = pressure of the atmosphere or of the exterior medium,

then by applying the momentum principle to fluids, we get the following equation:

$$F_i = v_2 m + (p_2 - p_3) A_2 = v_2 \frac{\dot{w}}{g} (p_2 - p_3) A_3 \qquad (9)$$

The term $(p_2 - p_3)A2$ is the thrust which acts in the same direction and on the same outlet area A_2 of the exhaust nozzle.

Also, V_2 equals V_1, the exit velocity of the exhaust stream.

To find the effects of thrust at various altitudes, let us compare the thrust with the pressure p_1 inside the combustion chamber.

By applying the principle of the conservation of matter to the gas stream of the exhaust nozzle of the jet propulsion motor, we get:

$$w_1 = w_2 = \frac{A_g \; v_g}{V_g} = \frac{A_2 \; v_2}{V_2} \qquad (10)$$

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in which A_g is the throat or most constricted section of the exhaust nozzle,

v_g is the gas flow velocity in the throat, and

V_g is the specific volume in cubic feet per pound in the throat.

If we substitute the values in equation (9), we get:

$$F_i = \frac{v_2}{g} \frac{A_g \; v_g}{V_g} + (p_2 - p_3) A_2 \tag{11}$$

From equation (11) the ideal thrust is determined, and by applying an experimental or empirical factor Δ, we obtain the effective thrust.

If F_e equals Δ F_i and

if we call

Ag the throat area,

C_F the thrust efficient,

p_1 the pressure inside the throat, then we have

$$F_e = F = \Delta C_F \, p_1 \, A_g \quad . \tag{12}$$

The connection factor varies between 0.92 and 1 and depends on the situation in the exhaust nozzle.

With respect to the thrust coefficient, by association of equations (11) and (12) and applying the principles of thermodynamics, which we shall not go into here, the following value is obtained:

$$C_F = \sqrt{\frac{2\,k^2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]} + \frac{p_2 - p_3}{p_1}\left(\frac{A_2}{A_g}\right) \tag{13}$$

In this equation we note that: p_1 is the pressure inside the combustion chamber; p_2 is the pressure of the combustion gases at the exhaust nozzle outlet; and p_3 is the atmospheric pressure of the surrounding medium.

A good exhaust nozzle must be designed to give a straight jet of gases. This will make $p_2 = p_3$ or the exit gas pressure almost equal to atmospheric pressure, but this can only be accomplished within certain limits since during flight p_3 varies according to the altitude. Since p_3 tends to decrease in the atmospheric layer there is greater expansion in the exhaust nozzle and therefore the thrust increases; that is, when p_2 decreases, C_F increases.

This means that the thrust $F_e - F = \Delta C_F p_1 A_g$ -- varies in proportion to the C_F.

Figure 3 shows graphically how the atmospheric pressure varies at different altitudes above sea level.

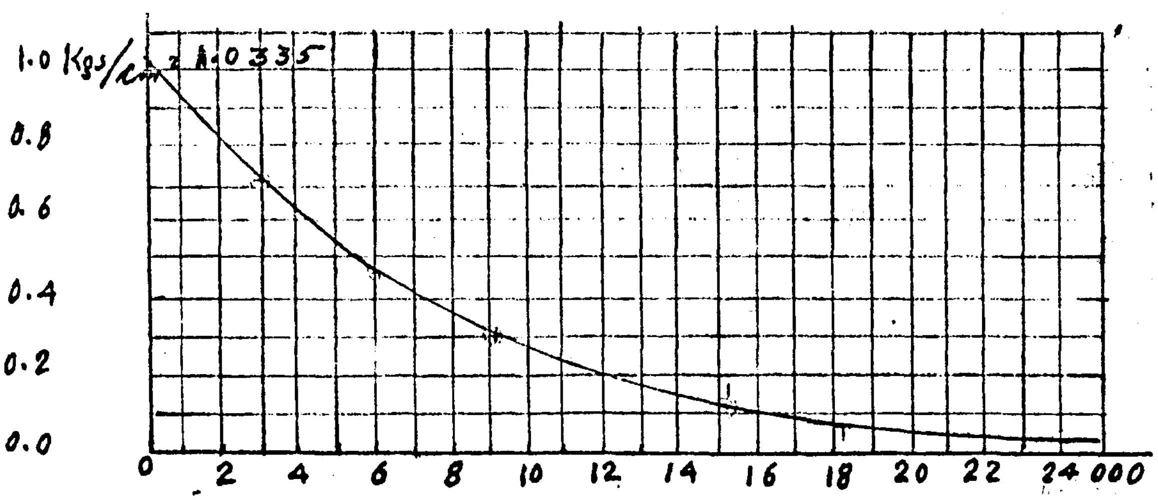

Fig. 3. Variation in atmospheric pressure in relation to altitude above sea level.

Pressure p_1 in the combustion chamber of liquid-fuel engines remains constant since the injection system is designed for automatic control, that is, an equal amount of fuel is used at any given time. This is not the case for solid-fuel rockets since the entire space occupied by the solid fuel is the combustion chamber. Therefore, calculations must be made within definable limits.

CHAPTER III

Gravitational Force

This force, as previously stated, is designated as W; it is due to the earth's attraction and is equal to M x g.

M is the mass at the moment when the analysis is undertaken; g is the gravitational acceleration which, in our aerodynamic study of any rocket, is the earth's resistance to anything leaving its surface.

Mass M will have the value of M_o before the rocket's combustion is initiated; it will be m after combustion of the propellants or fuel (m_p). We gave the variation of this mass in equation 3. $\frac{dM}{dt} = x\, m_p$.

The gravitational acceleration g varies with altitude; at sea level g = 9.8066 in the latitude of Ecuador where the earth's radius R_o = 6.387 kilometers. (In the US system g_o = 32.2 feet/sec^2, and R_o = 3963 miles.)

In order to determine the acceleration due to gravity at any height "h" above sea level, we start with the known principle that it varies with the square of the distance from the center of the earth. If g_o is the acceleration on the surface of the earth with a radius of R_o, then g at height "h" will have the value:

$$g = g_o \left(\frac{R_o}{R}\right)^2 = g_o \left(\frac{R_o}{R_o + h}\right)^2 \qquad (14)$$

CHAPTER IV

Air Resistance

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The force of the air's resistance, which is called "drag" in English, is the force which opposes the free movement of a rocket within the atmosphere; it is found according to the formula:

$$D = 1/2 \rho \, S \, C_D \, V^2 \qquad (15)$$

in which ρ is the density of the air,
S is the largest cross section of the rocket
C_D the retardation or drag coefficient, and
V the flight velocity of the rocket.

Equation (15) is the classical one expressing aerodynamic resistance but only when the vehicle moves vertically in the atmosphere. It gives the resistance which depends on: the density of atmospheric air which varies with height above sea level; the maximum cross section of the rocket; the square of the velocity with which the rocket changes its position; and the drag coefficient which is related to the Mach number during flight.

The drag coefficient C_D encompasses all the resistance effects produced in the nose cone, the external surface friction of the rocket, and in the rocket's tail assembly. Without going into too much detail, we can give a summary explanation of each resistance effect, since our main purpose is solving the flight equation.

The resistance effect of drag in the nose cone is due to the air slowing down in the vicinity of the rocket's nose cone. In that area, it produces much greater pressures than those of the surrounding atmosphere and very high temperatures result which force us to use materials there that are resistant to high temperatures. Air slippage of the dividing boundary layer of the atmosphere surrounding the rocket produces external surface friction. The resistant drag effect at the

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tail is caused by the separation of the air stream at that point, creating a reverse-acting suction.

The effect at the nose cone and that on the rocket's exterior surface are more difficult to evaluate; nevertheless, the jet action has a great influence on the tail drag, because during the time the jet is blasting, there is no suction. From this we can say that there must always be propulsion while inside the atmosphere.

At high supersonic speeds the drag at the nose is much greater than all the friction effects due to surface and tail drag. This means that the nose must be kept at the proper angle to decrease resistance.

As for the drag coefficient C_D, various books and pamphlets have been published with graphs showing how it varies in relation to the Mach number. These data were used in aerodynamic calculations on liquid-fuel rockets SCT-1 and SCT-2, as well as on the solid-fuel rocket SCT-S5. In the graph shown as Fig. 4, the C_D values are given for rockets, for a rocket shaped like the V2 bomb, and for cylindrical-shaped rockets.

In Fig. 5 we can see how the density of air varies at different altitudes above sea level.

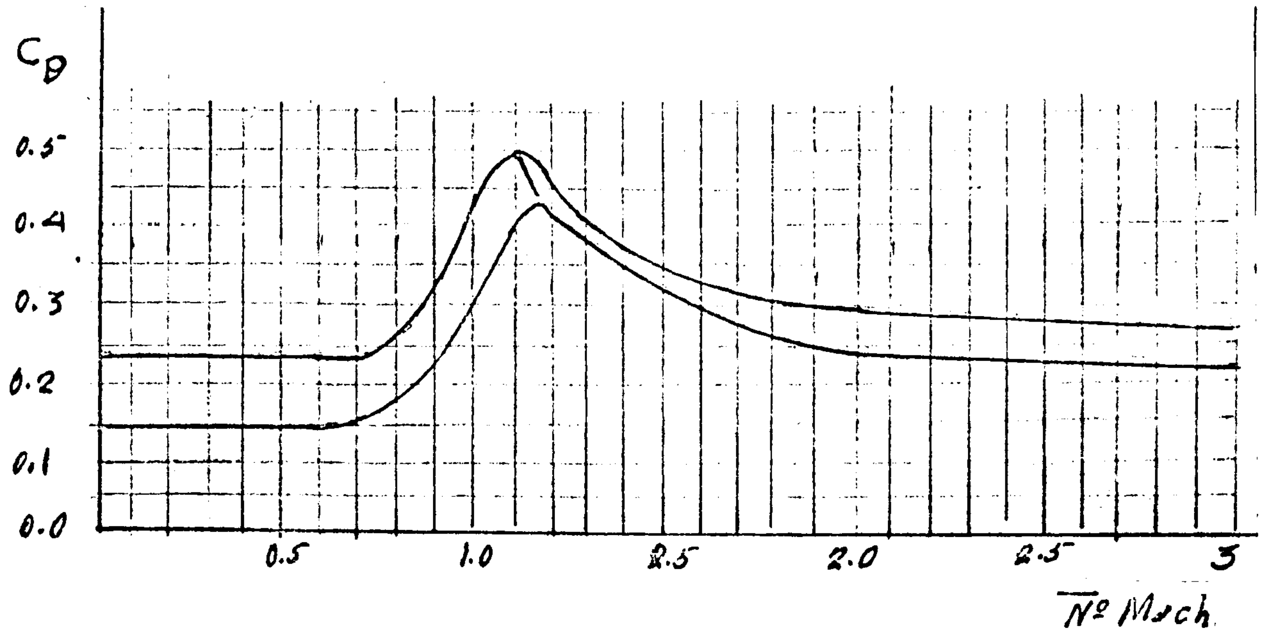

Fig. 4 The Drag Coefficient compared with the Mach Number.

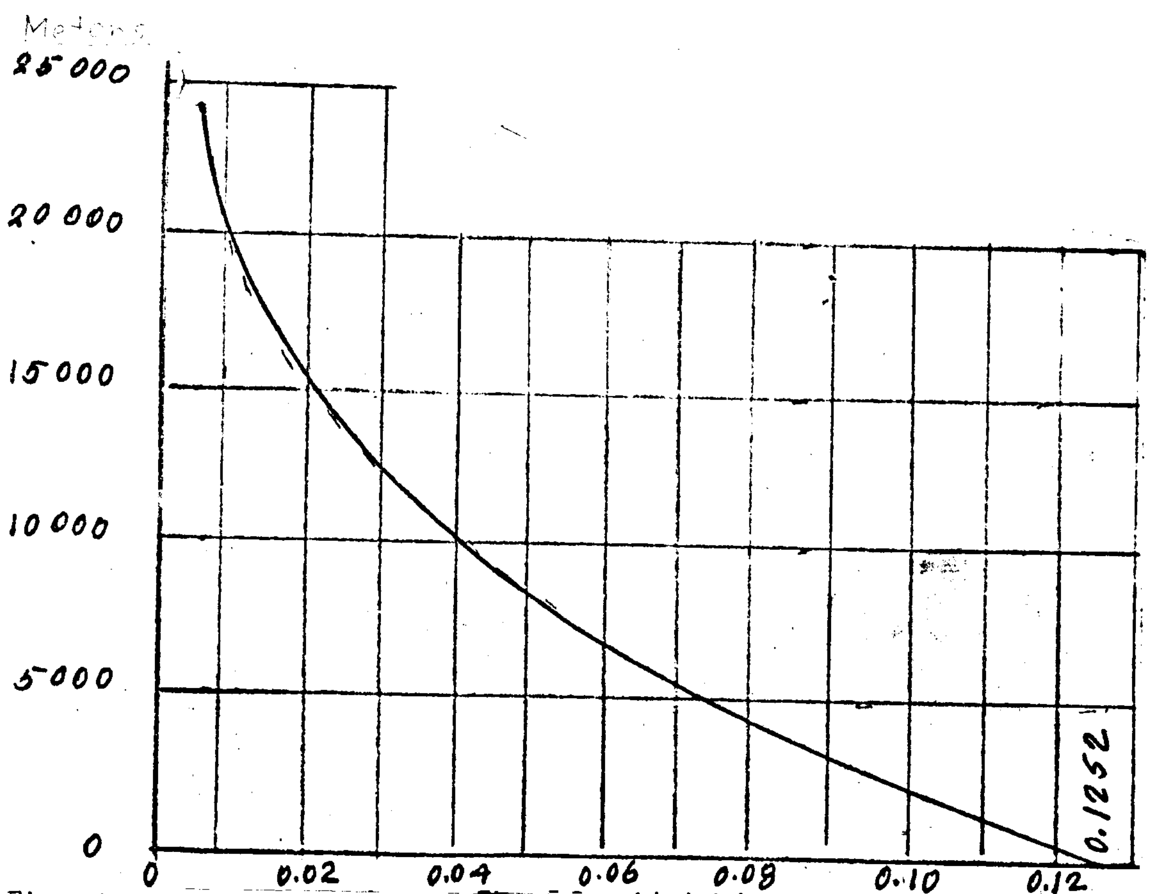

Fig. 5 Air Density at Various Altitudes Above Sea Level.

CHAPTER V

Vertical Flight Without Air Resistance

The procedure for resolving the aerodynamics of a rocket's vertical flight is based on an analysis of the nature of flight both with and without air resistance.

For flight without air resistance, equation (1) may be written as:

$$F_e = Ma = F - W = M\frac{dv}{dt}$$

Therefore, the instantaneous velocity is:

$$v = \int \left(\frac{F - W}{M}\right) dt$$

Making the necessary substitutions, we get:

$$v = \int \frac{\frac{W}{g} V_j - Mg}{M} dt = \int \left[\frac{W}{g} \cdot \frac{V_j}{M} - g\right] dt \qquad v = \int \left[x\, m_p \frac{V_j}{M_o - x\, m_p t} - g\right] dt$$

If x is this share of m_p which burns per second, then we may write $x = 1/t_c$ in which t_c is the cut-off time or the time required for the combustion of the mass m_p; if we integrate this last equation we get

$$v = V_j \log_e \left[1 - \frac{m_p}{M_o\, t_c} t\right] - gt + V_o \qquad (16)$$

in which v is the ideal velocity without air resistance at any moment within the combustion time and in which V_o is the starting velocity when the combustion engendered speed in the rocket.

The velocity reached once combustion takes place, occurs when $t = t_c$ so that we may write the following equation:

$$v_c = V_j \log_e \left(1 - \frac{m_p}{M_o}\right) - gt_c + V_o \qquad (17)$$

The height which the rocket could reach without encountering any air resistance would be:

$$v = \frac{dh}{dt} = -V_i \ \text{Log}_e \left[1 - \frac{m_p}{M_o \ t_c} t \right] - gt + V_o$$

$$h = -V_i \int \text{Log}_e \left[1 - \frac{m_p}{M_o \ t_c} t \right] dt - g \int t\,dt + V_o \int dt$$

By integration we obtain the height at any moment during which the propulsion or combustion is still going on. We call this height h_p.

$$h_p = \left[\frac{V_i \ Motc}{m_p} - V_i t \right] \text{Log}_e \left[1 - \frac{m_p \ t}{M_o \ t_c} \right] + V_i t - \frac{1}{2} g t^2 + V_o t + h_o \qquad (18)$$

In this equation, h_o is the height which the rocket attains when combustion starts; the maximum height is h_c which is reached when the combustion terminates. The maximum height is attained when $t = t_c$ which is the cut-off time or time when the combustion ceases.

$$h_c = t_c \left[\frac{V_i \ M_o}{m_p} - V_i \right] \text{Log}_e \left(1 - \frac{m_p}{M_o} \right) + V_i \ t_c - \frac{1}{2} g \ t_c^2 + Vot_c + h_o \qquad (19)$$

Without taking into consideration the resistance of air, only gravity can slow down the rocket.

In the dynamics of firing missiles of a certain velocity and which are ejected vertically, the missiles travel in accordance with the formula $h = \frac{v^2}{2g}$. In our case, when combustion ceases the rocket is regarded at that time as a missile fired at speed V_c; gravity slows it down until velocity 0 is attained. In this period of time, the rocket goes to height h_g.

Without air resistance, the rocket reaches a maximum altitude of:

$$h_{max} = h_c + h_g \qquad (20).$$

having the following value: $hg = \frac{V_c^{\,2}}{2\,g}$ (21)

If we substitute the values in equation (20) and simplify it, then we get the following formula for maximum height:

$$h_{max} = \frac{V_j \; t_c}{\frac{m_p}{M_o}} \; Log_e \left(1 - \frac{m_p}{M_o}\right) + \frac{V_j^{\,2}}{2g}\left[Log_e \left(1 - \frac{m_p}{M_o}\right)\right]^2 + V_j \; t_c$$

which we can further simplify by making $m_p = M_o - m$

$$h_{max} = \frac{V_j \; t_c}{\left(1 - \frac{m}{M_o}\right)} \; Log_e \frac{m}{M_o} + \frac{V_j^{\,2}}{2g}\left[Log_e \frac{m}{M_o}\right]^2 + V_j \; t_c \tag{22}$$

The maximum desirable height to be reached will always depend on the V_j, the exit velocity of the combustion gases which is directly related to the duration of the combustion and in this study is called cut-off time t_c; when m_p/M_o has maximum values greater heights are reached. It must be constantly borne in mind that m_p/M_o approaches unity, that is the rocket must be lightweight but provide great capacity for propellants (m_p).

CHAPTER VI

Vertical Flight with Air Resistance

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When analyzing the air resistance, its retardation effect on the motion is indicated in equation (1) and a substitution of values is made.

$$F_e = Ma = F - W - D = M\frac{dv}{dt} \quad \therefore \quad dv = \frac{F - W - D}{M}dt$$

The velocity of the rocket in the atmosphere will be:

$$v = \int \frac{x\, m_p\, V_j}{M_o - x\, m_p t}dt - \int g\,dt - \int \frac{\rho v^2\, S\, C_D}{2(M_o - x\, m_p t)}dt$$

If we simplify this equation and show the combustion time:

$$x = \frac{1}{t_c}\Big| y \quad \text{and also} \quad y$$

$$v = \frac{dh}{dt} = \int_0^t \frac{y\, V_j}{t_c - yt}dt - \int_0^t g\,dt - \int_0^t \frac{\frac{C_D S}{M_o} \cdot \frac{1}{2}\rho v^2}{1 - y\frac{t}{t_c}}dt + V_o$$

In the last equation V_o is the launching velocity before combustion starts, that is, the velocity of the rocket when combustion starts. In the equation, C_D is a variable because with changing height and velocity the Mach number increases and this accounts for C_D having different values as seen in Figure 4. The same is true for the density of air ρ; this obliges us to integrate for finite times during which C_D and ρ are considered constants since they vary only little in value. We assign an average value to g, which is the gravitational acceleration in the atmosphere.

With regard to the integration in the foregoing equation for obtaining the maximum velocity during propulsion (or what we have been calling v_c), its value is determined by integration within the limits of 0 and t_c; and by substituting m_p/M_o for y, we get:

$$v_c = -V_j \ \text{Log}_e \left(\frac{M_o - m_p}{M_o}\right) - g\ t_c - \frac{C_D S}{M_o}\int_0^{t_c} \frac{\frac{1}{2}\rho\, v^2}{1 - y\frac{t}{t_c}}\,dt$$

The value of $\left.\int_0^{t_c}\frac{\frac{1}{2}\rho\, v^2}{1 - y\frac{t}{t_c}}\right|$

is obtained by a graphic integration, and values are given for velocity v. All textbooks on aerodynamics choose a certain letter for it and we indicate it as B_1; thus, the value of the cut-off velocity will be:

$$v_c = -V_j \ \text{Log}_e \left(\frac{M_o - m_p}{M_o}\right) - g\ t_c - B_1 \frac{C_D S}{M_o} + V_o \qquad (23)$$

The cut-off height, or the height attained when combustion ceases, will be:

$$h_c = V_j\ t_c\left[1 - \left(\frac{M_o - m_p}{m_p}\right)\text{Log}_e \frac{(M_o - m_p)}{M_o}\right] - \frac{1}{2} g\ t_c^2 + V_o\ t_c$$

$$+ h_o - B_2 \frac{C_D S}{M_o} \qquad (24)$$

in which $B_2 = \int_0^{t_c} B_1\,dt$

The maximum height reached is determined as was done before, with the exception that we now take into account the air resistance. After combustion ceases, the rocket continues to fly like a missile with a starting velocity of v_c; due to inertia it will reach height $h_g = \frac{v_c^2}{2g}$ and therefore the maximum height to which it rises is:

$$h_{max} = h_c + h_g = \frac{V_i^2}{2g}\left[\text{Log}_e \frac{M_o - m_p}{M_o}\right] + {}^2 \; V_i \; t_c \left[1 + \frac{M_o}{m_p} \text{Log}_e \frac{(M_o - m_p)}{M_o}\right] + \frac{V_o^2}{2g} + h_o - \frac{V_o}{g} V_i \; \text{Log}_e \frac{(M_o - m_p)}{M_o} - \frac{C_D S}{M_o}\left[B_2 + B_1 \frac{V_o - V_i}{g} \text{Log} \frac{M_o - m_p}{M_o} - B_1 \; t_c - B_1^2 \frac{C_D S}{2g \, M_o}\right] \quad (25)$$

When great heights are reached the resistance due to air can be ignored. Then equation (25) will become simplified; then, too, we may consider V_o and also h_o as being zero, which is usually the case when launchings are made from the earth. The solution to equation (25) is quite complicated and more so when C_D and ρ vary with the altitude attained. All these very complicated details led us to solve the equation of flight by another procedure without loosing sight of the dynamics of the problem.

We had to present the mathematical development outlined above so that the reader would better understand this "graphical-mathematical" method of solving the equation.

CHAPTER VII

Solution of the Equation for Vertical Flight by the Graphical-Mathematical Method

1. Climbing with Propulsion

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The solution of the dynamic equation of vertical flight was prompted by the desire to obtain a method which allows us to analyze graphically and mathematically the change of the ideal velocity v_i (without air resistance) as well as the effective velocity v_e (bearing in mind the air resistance) with respect to time t. From the plot, in Fig. 6, subsequent values of the effective velocity may be derived.

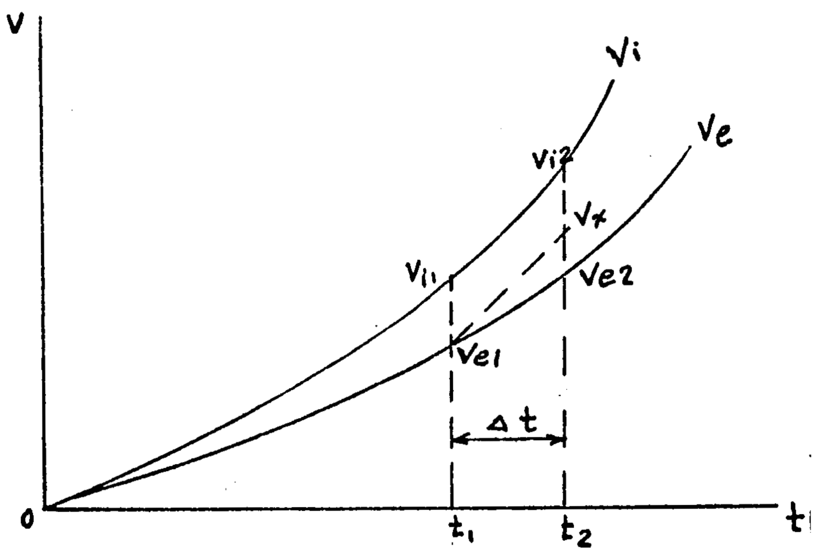


Fig. 6 The Velocity-Time Curves during Combustion.

The ideal velocity when there is no air resistance is given as follows:

$$v_i = \int \frac{F - W}{M} dt$$

The effective velocity is the real one in the atmosphere since it take into account air resistance. It is determined from the equation:

$$v_e = \int \frac{F - W - D}{M} dt$$

At a given time t, the loss of velocity due to air resistance is found from the difference between velocities at the ideal and actually existing conditions:

In the analysis to be undertaken, graphs for the "$v_i t$" and the "$v_e t$" are plotted at the smallest possible measurements so that we may consider the effect of the drag coefficient C_D according to the Mach numbers obtained. If we assume that within such finite limits the drag force is constant, we may write:

(26)

The integral $\int \frac{dt}{M}$ is a modulus or parameter which varies as the function of time because M is the mass in the moment the analysis is undertaken. We call this modulus Q_t and we may therefore write it:

If we substitute $y = m_p/M_o$ and integrate, we get:

$$Q_t = - \frac{t_c}{m_p} \operatorname{Log}_e \left(1 - \frac{m_p}{M_o\, t_c} t\right)$$

(27)

$$Q_t = - \frac{t_c}{y M_o} \operatorname{Log}_e \left(1 - \frac{yt}{t_c}\right)$$

By simplifying and substituting $\phi = \frac{m_p}{t_c}$

we get:

$$Q_t = \frac{1}{\phi} \operatorname{Log}_e\left(\frac{M_o}{M_o - \phi t}\right) \qquad (27)$$

The loss of velocity during a finite interval is:

$$v_i - v_e = DQ_t = \frac{Dt_c}{yM_o} \operatorname{Log}_e\left(1 - \frac{yt}{t_c}\right) = \frac{D}{\phi} \operatorname{Log}\left(\frac{M_o}{M_o - \phi t}\right) \qquad (28)$$

If we undertake in equation (15) the substitution:

$$K_1 - \frac{1}{2} \rho \; S \; C_D \qquad (29)$$

then we get for the drag force:

$$D = K_1 \; v^2 \qquad (30)$$

In Fig. 6 the ordinates are shown as intersecting the curves of the ideal velocity (v_i) and of the effective velocity (v_e) during the finite period of time Δt lying between time t_2 and time t_1. We assume that the differences for the air densities and the drag coefficients C_D for times t_1 and t_2 could be ignored.

In the plot of figure 6 we have drawn a parallel of $\overline{v_{e1} - v_x}$ to $\overline{v_{i1} - v_{i2}}$ for the purpose of stating a relationship with the subsequent velocity for t_2 .

Between times t_1 and t_2 we assume that there is an average drag force D, and we assume that only the velocity changes between v_{e1} and v_{e2} ; therefore, we get:

$$D = K_1 \left(\frac{v_{e2} + v_{e1}}{2}\right)^2 = \frac{K_1}{4} (v_{e2} + v_{e1})^2 \qquad (31)$$

From plot 6 for time t_2, we get:

$$v_{i2} - v_{e2} = (v_{i1} - v_{e1}) + (v_x - v_{e2}) = DQ_2$$

If we substitute values and simplify, we get:

$$(v_{i1} + v_x) - (v_{e2} + v_{ei}) = \frac{K_1 Q_2}{4} \quad (v_{e2} + v_{e1})^2$$

This can be rearranged to show that we have to deal with a quadratic equation, or one of the second degree:

$$\frac{K_1 Q_2}{4} (v_{e2} + v_{e1})^2 - (v_{e2} + v_{e1}) - (v_{i1} + v_x) = 0$$

This means that the effective velocity at time t_2 is:

$$v_{e2} = \frac{-1 \pm \sqrt{1 + K_1 \; Q_2 \; (v_{i1} + v_x}}{\frac{K_1 \; Q_2}{2}} - v_{e1} \qquad (32)$$

On the basis of equation (32), in order to find the next or subsequent velocity value v_{e2} we must plot the ideal velocity-time curve so that we can dra[illegible] parallels $\frac{\quad\quad}{v_{i1} - v_x}$ to then determine the value of v_x. Using th[illegible] in equation (32), we can determine the effective velocity of the rocket at time t_2. Then between t_1 and t_2 we calculate the values of K_1 and Q_2 because K_1 will vary according to height and velocity. We calculate K_1 in point t_1 and Q_2 in point t_2. In this way, we are always realistic; in case more exact calculations are required, it would be necessary to repeat the calculations in each point to determine h_2, and to determine more correctly and to find a first velocity v_{e2} for the Mach number which furnishes the value of the drag coefficient. Close to the sound barrier where the Mach number is unity, we will get the greatest drag coefficient.

The effective height reached is determined with the average values of v_{e1} and v_{e2}, derived from the respective plots.

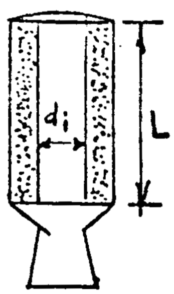
When a liquid-fuel rocket is analyzed, the amount of fuel burned per second is constant, but for a solid-fuel rocket the flight analysis is more complicated because the propelling force varies between a minimum and a maximum. This means that the thrust varies as a function of time (its change with respect to time is determined in a test stand) since the exit velocity of the gases v_j also varies.

For the analysis of a solid-fuel rocket during the time of flight, we must write up a table with convenient time intervals, and we must when doing this be duly aware of the effect of the sound barrier, i.e, when the Mach number is equal to 1.

Equation (32) encompasses the dynamic effect during combustion; if we determine this we will realize that the behaviour of the rocket is different. When using this equation, it is always necessary to follow the accumulative system of the tables since it must be used between two points i.e., the M_o must be calculated when it is used again.

In a solid-fuel rocket according to equation (27-B):

$$Q_t = \frac{1}{\phi} \operatorname{Log}_e \left(\frac{M_o}{M_o - \phi t} \right); \quad \phi = \frac{m_p}{t_c} \quad \text{is variable.}$$



If L = the length of the solid-fuel cylinder,

d_i = the internal diameter of the core before the rocket is ignited,

d = the diameter of the core at any given time

v_{cl} = the linear radial combustion velocity of the solid fuel,

β = the specific weight of the solid fuel,

then we may write for the mass of fuel burned up at any given time:

$$\phi = \frac{\pi d L v_c \beta}{g}$$

which means that the instantaneous mass of the rocket is:

$$M = M_o - \phi t = M_o - \frac{\pi d L v_c \beta}{g}$$

When combustion ends, the propulsion force is F = 0; thereafter the rocket continues to rise due to inertia, but it has to overcome two resistant forces: gravity and air resistance. It will stop rising when the effective flight velocity is $v_e = 0$.

Let us now analyze flight conditions without propulsion, i.e., for F= 0. /24

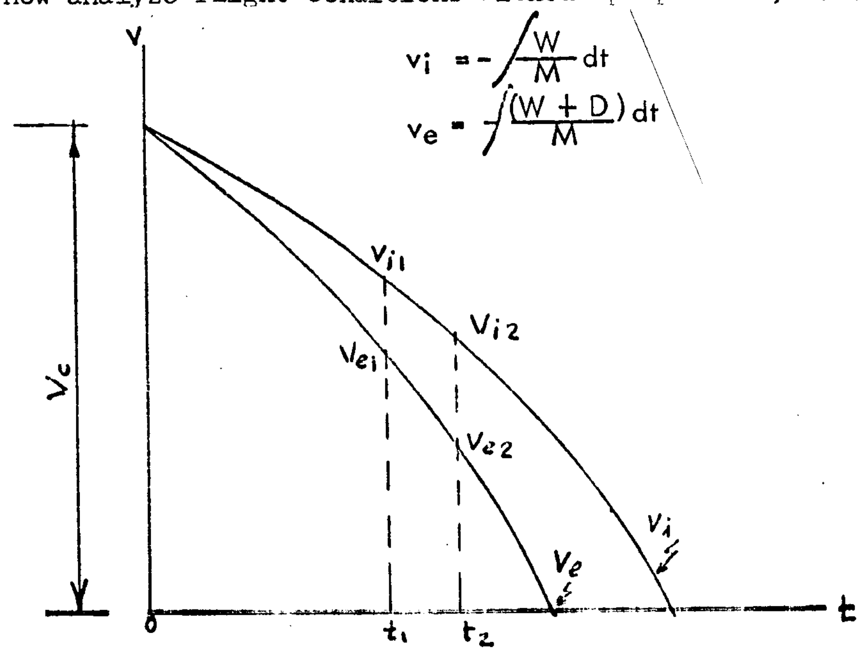


Fig. 7. The velocity-time curve for ascent without propulsion.

Now the rocket no longer is carrying fuel, $m_p = 0$

$$M_0 = m_p + m = m$$

The mass is then constant and at any time its value is m which is dead weight. Under these conditions, also, the rocket's initial velocity is v_c, the cut-off velocity. So, for any given time, we may write:

$$v_i - v_e = \frac{D}{m} t \tag{33}$$

When the rocket continues to climb to high altitudes due to inertia, the value of air resistance D grows smaller and smaller because the air density has a limit value of zero and the effective velocity will drop to zero. Thus, when calculating the value D, we shall not operate with an averagevelocity but shall take the velocity of the next point.

Substituting the values in equation (33) and applying it in point t_1, we get:

$$D = K_1 {v_{e2}}^2$$

$$v_{i2} - v_{e2} = \frac{K_1 (v_{e2})^2}{m} t \quad ó \quad \frac{K_1}{m} t \; (v_{e2})^2 + (v_{e2}) - v_{i2} = 0$$

The foregoing is a second-degree equation and its solution is:

$$v_{e2} = \frac{-1 \pm \sqrt{1 + \frac{4K_1 t}{m} v_{i2}}}{\frac{2K_1}{m} t} \qquad (34)$$

in which:

$$K_1 = \frac{1}{2} \rho C_D S$$

Equation (34) is valid when $v_c = v_i = v_e$.

Accordingly, we calculate the values of the effective velocity "v_e"; altitudes or heights "h" are determined. From these new altitudes we obtain new values for air density.

A table, set up in columns, will facilitate plotting the velocity-time graphs; inside the atmosphere we assume that the acceleration due to gravity (g) is constant. There is no doubt that in high altitudes, with high power rockets of long propulsion periods, the effect of atmosphere becomes secondary; liquid-fuel rockets do not attain the greatest velocities within the atmosphere.

Now using equation (34), we analyze the flight until the highest altitude due to inertia is attained. At that maximum altitude, theeffective and ideal velocity reaches zero (0).

Let us first simplify equation (34).

If we integrate the equation for the ideal velocity and keep in mind that $v_i = v_e$ when $t = 0$, we get:

$$v_{i2} = v_c - gt \qquad (35)$$

$$v_{e2} = \frac{-1 \pm \sqrt{1 + \frac{4K_1}{m}(v_c - gt) \cdot t}}{\frac{2K_1}{m}t} \qquad (36)$$

3. The Rocket Falls into the Atmosphere

The aerodynamic analysis starts when h is at its maximum and when v_i and $v_e = 0$; the initial velocity is zero but when descent begins, at any instant only external forces will act; i.e., gravity will move the rocket and the drag or air resistance will oppose the falling motion. We may thus write:

$$F = Ma = \frac{dv}{dt} = W - D$$

For the rocket's descent, the ideal velocity without air resistance:

$$v_i = \int \frac{W}{M} dt$$

The effective velocity or that with air resistance is expressed:

$$v_e = \int \frac{(W - D)}{M} dt$$

In both cases, mass $M = m$ is constant, which simplifies analysis.

The loss of velocity due to air resistance is seen as:

$$v_i - ve = \frac{D}{m} t \qquad (37)$$

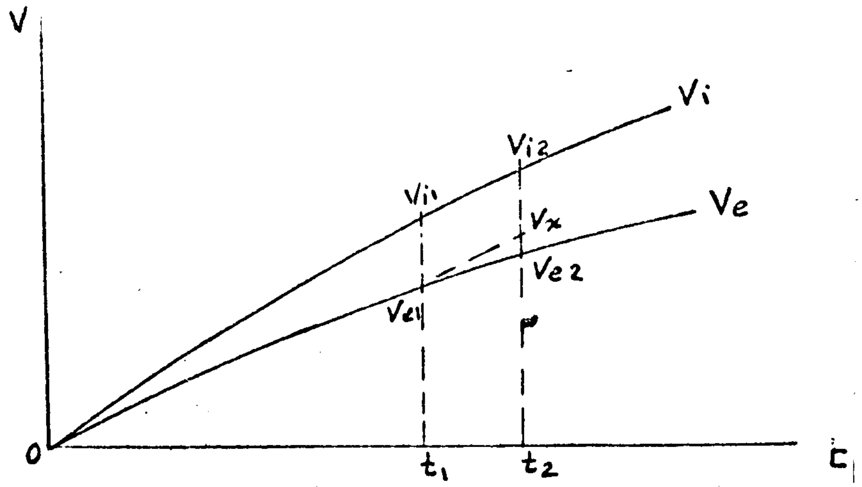


Fig. 8. The Velocity-Time Blot for Descent without Propulsion.

During time t_2 the decrease in velocity is -- if we accept that in the graph $(v_{e1} - v_x)$ is parallel to $(v_{i1} - v_{i2})$;

$$v_{i2} - v_{e2} = (v_{i2} - v_x) + (v_x - v_{e2})$$

$$= (v_{i1} - v_{e1}) + (v_x - v_{e2}) = \frac{K_1 t_2}{4m} \quad (v_{ei} + v_{e2})^2$$

$$\frac{K_1 t_2}{4m}(v_{e1} + v_{e2})^2 + (v_{ei} + v_{e2}) - (v_{i1} + v_x) = 0$$

and therefore the velocity at time t_2 is:

$$v_{e2} = \frac{-L \pm \sqrt{1 + \frac{K_1 t_2}{m}(v_{i1} + v_x)}}{\frac{K_1 t_2}{2m}} - v_{e1} \qquad (38)$$

On the basis of equation (38), flight graphs may be plotted; accordingly, as the effective velocities are calculated, the altitudes are determined, as *relatively* new values are obtained: ρ, the air density, the Mach number, etc. If more precise calculations are desired, after the v_{e2} and the h2 are determined, then we may give k_1 a new value since in this case it is practically k_2. By using a table with as many columns as necessary, results are gathered which indicate how the rocket will move aerodynamically during its return to earth.

Presented here is the graphical-mathematical theory of how a rocket will behave inside the atmosphere. In the atmosphere above supersonic speeds, temperatures above 350° C are reached at the rocket's nose. This forces us to use materials highly resistant to those high temperatures.

Space rockets traveling inside the atmosphere have low velocities in comparison with the velocities required to put vehicles into orbit. In theoretical studies on this topic, air resistance has been minimized, but the cost of the fuel to overcome that resistance cannot be ignored.

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k /s/

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